

Positroids, Plabic Graphs, & Scattering Amplitudes in MATHEMATICA

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ABSTRACT: The many intricate connections between scattering amplitudes, on-shell diagrams, and the positroid stratification of the Grassmannian has recently been described in detail. In order to facilitate the exploration of this rich correspondence, we have prepared a public MATHEMATICA package called “**positroids**” which includes an array of useful tools including those for the construction of canonical coordinates for positroid configurations, the drawing of representative on-shell (plabic) graphs, and the evaluation of on-shell differential forms. This note documents the functions made available by the **positroids** package; the package’s source code together with a MATHEMATICA notebook containing many detailed examples of its functionality are included with this note’s submission files on the **arXiv**.

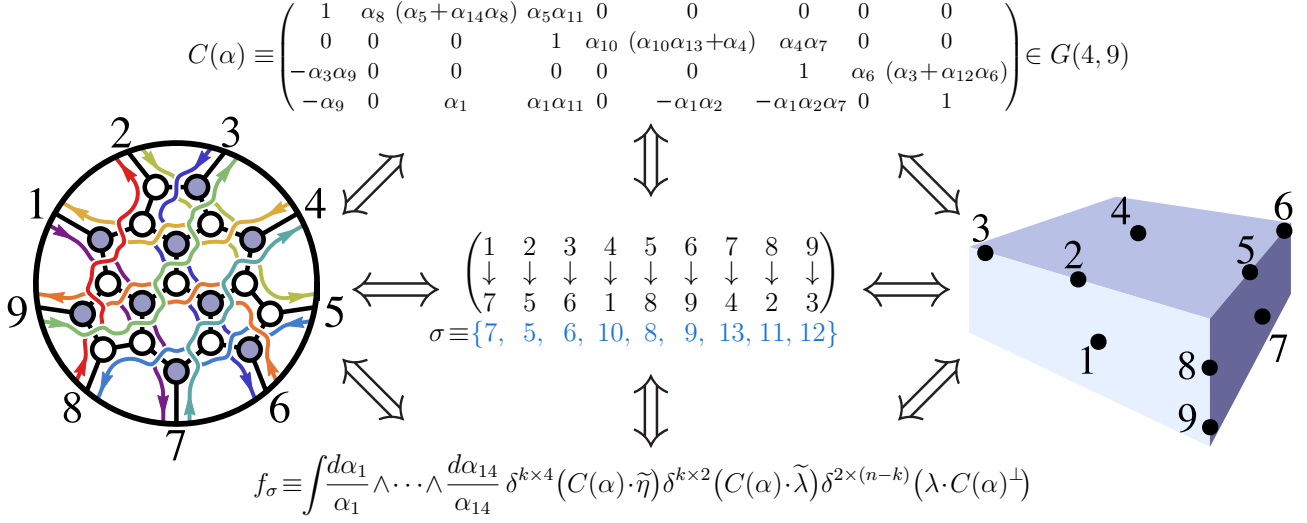
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1. Introduction

The recent work of [1] presents a comprehensive summary of the extensive correspondence between “on-shell diagrams”, [2–6], the Grassmannian contour integral described in [7–11], scattering amplitudes in planar, maximally supersymmetric (“ $\mathcal{N} = 4$ ”) Yang-Mills (SYM), [12–18], and the combinatorics and geometry of what is known as the *positroid* stratification of the Grassmannian, [19, 20]. At the heart of this story is the fact that scattering amplitudes can be represented (to all loop orders) in terms of on-shell diagrams, and that (reduced) on-shell diagrams can be fully characterized *combinatorially* by *permutations*—associated with left-right paths; moreover, these same permutations label the configurations of the positroid stratification of the Grassmannian $G(k, n)$ of k -planes in n dimensions. These strata are naturally endowed with *positive* coordinates α_i and a canonical volume-form, [21, 22], which when expressed in terms of positive coordinates, is simply: $d \log(\alpha_1) \wedge \cdots \wedge d \log(\alpha_d)$. Because on-shell diagrams can be *directly represented* (and computed) as integrals of this invariant volume-form over their corresponding positroid configurations, this makes the evaluation of on-shell diagrams exceedingly simple.

We can illustrate this rich correspondence with the following example:



Starting with the on-shell diagram on the left, we find that it would be labeled (via left-right paths) by the permutation denoted $\sigma \equiv \{7, 5, 6, 10, 8, 9, 13, 11, 12\}$; this permutation *also* labels the Grassmannian configuration drawn (projectively¹) on the right—a configuration represented by the matrix $C(\alpha)$ above, parameterized by the positive coordinates α_i ; in terms of $C(\alpha)$, the corresponding on-shell ‘function’ f_σ associated with the diagram is determined by the integral at the bottom.

We will not review these ideas here, but instead refer the interested reader to reference [1] for a thorough introduction and summary together with a more comprehensive list of references to the existing literature.

In order to help facilitate further investigation along these lines, however, we have prepared a MATHEMATICA package called “**positroids**”—which is documented in this note. The **positroids** package makes available many of the essential tools required to investigate the myriad connections between on-shell physics, scattering amplitudes, and the combinatorics and geometry of the positroid stratification of the Grassmannian.

¹Here, each dot represents a column of the matrix $C(\alpha)$ viewed projectively as a point in \mathbb{P}^3 .

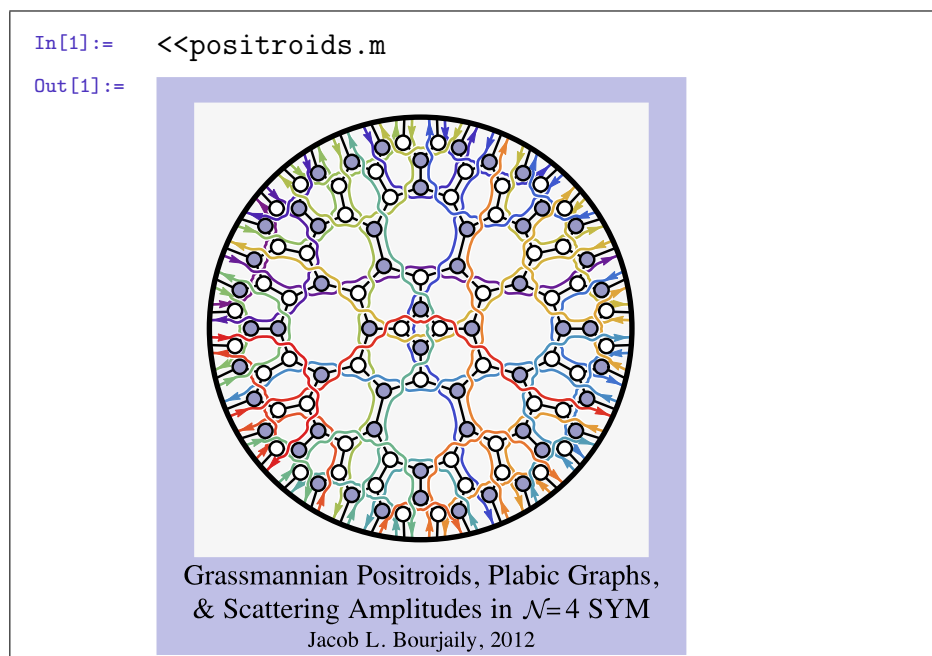
2. The MATHEMATICA Package positroids

2.1 Obtaining the positroids Package and Demonstration Notebook

From the abstract page for this paper on the [arXiv](#), follow the link “other formats” (below the option for “PDF”) and download the “source” for the submission. The source-file will contain² the `positroids` package’s main source-code (`positroids.m`), together with a demonstration notebook (`positroids_package_demo.nb`) which describes with detailed examples many of the functions defined by the package.

2.2 Using the positroids Package

Upon obtaining the package, users should open and evaluate the MATHEMATICA notebook `positroids_package_demo.nb`; this notebook will copy `positroids.m` to the user’s Application directory for MATHEMATICA; this will allow `positroids` to be started in any future notebook via the command:



(If the file “`positroids.m`” has not yet been copied to the user’s Application directory, then the package can be initialized by saving a notebook to the directory where `positroids.m` is located, and evaluating “`SetDirectory[NotebookDirectory[]]`” prior to the command “`<<positroids.m`”.)

²Occasionally, the “source” file downloaded from the [arXiv](#) is saved without any extension; this can be ameliorated by manually appending “`.tar.gz`” to the name of the downloaded file.

3. Functions Defined by positroids

3.1 Operations on Permutations Labeling Positroids

- `boundary[permutation_]`: returns a list of permutation labels for positroid cells in the boundary, ∂ , of the cell labeled by `permutation`. For example, the co-dimension one boundaries of the generic configuration in $G_+(3,6)$ are:

```
In[1]:= boundary[{4,5,6,7,8,9}]
Out[1]:= {{5,4,6,7,8,9},{4,6,5,7,8,9},{4,5,7,6,8,9},
          {4,5,6,8,7,9},{4,5,6,7,9,8},{3,5,6,7,8,10}}
```

- `checkOperator[permutation_][operator_]`: returns:
 - 0 if the minor (`operator`) vanishes for the cell labeled by `permutation`;
 - 1 if minor (`operator`) is *non*-vanishing for the cell labeled by `permutation`.

```
In[1]:= {checkOperator[{3,5,6,7,8,10}][{1,2,3}],
          checkOperator[{3,5,6,7,8,10}][{2,3,4}]}
Out[1]:= {0,1}
```

- `cyclicize[permutation_]`: returns a *sorted* list of non-repeating permutations in the same cyclic class as `permutation`.

```
In[1]:= cyclicize[{6,5,8,7,10,9,12,11}]
Out[1]:= {{4,7,6,9,8,11,10,13},{6,5,8,7,10,9,12,11}}
```

- `cyclicRep[permutation_]`: returns the lexicographically-first permutation in the cyclic-class of `permutation`.
- `decorate[permutation_]`: takes an ‘ordinary’ permutation σ of n integers, and returns a *decorated*, affine permutation $\hat{\sigma}$, adding n to the image of any a such that $\sigma(a) < a$. For example, applying `decorate` to the ordinary permutation associated with the on-shell diagram given in the introduction (section 1) gives:

```
In[1]:= decorate[{7,5,6,1,8,9,4,2,3}]
Out[1]:= {7,5,6,10,8,9,13,11,12}
```

- `dimension[permutation_]`: gives the dimension of the positroid stratum labeled by `permutation`; for example,

```
In[1]:= dimension[{3,5,7,6,8,14,10,12,11,13,16,21}]
Out[1]:= 20
```

- `dualGrassmannian[permutation_]`: if *permutation* labels a $(2n-4)$ -dimensional cell in the ‘momentum-space’ Grassmannian $\widehat{C} \in G(k+2, n)$, then `dualGrassmannian` returns the permutation label of the corresponding, $4k$ -dimensional cell in the ‘momentum-twistor’ Grassmannian $C \in G(k, n)$, and *vice-versa*; e.g.,

```
In[1]:= dualGrassmannian[{6,5,8,7,10,9,12,11}]
dualGrassmannian@%
Out[1]:= {2,5,4,7,6,9,8,11}
{6,5,8,7,10,9,12,11}
```

- `eulerCharacteristicTable[n_,k_]`: returns a table indicating the numbers of d -dimensional cells in the positroid stratification of $G_+(k, n)$. (This data is generated using the combinatorial results of [20].)

```
In[1]:= eulerCharacteristicTable[10,5]
Out[1]:=
```

dim	#	#	dim
24	1	10	25
22	55	220	23
20	715	2002	21
18	4985	11240	19
16	23210	44220	17
14	78087	128100	15
12	195315	276450	13
10	362175	437112	11
8	482670	482940	9
6	432060	339360	7
4	228102	126420	5
2	54600	16800	3
0	3150	252	1
	1865126	1865125	

Euler Characteristic: 1865126 – 1865125 = 1

- `intersectionNumber[permutation_,m_:4]`: returns the number of *isolated* solutions to m -dimensional kinematical constraints, $\Gamma^{\mathfrak{m}}(C)$, where C is the positroid labeled by *permutation*. Supposing that C is an (mk) -dimensional cell in the momentum-twistor Grassmannian³, `intersectionNumber` counts the number of

³`intersectionNumber` also tests $(2n-4)$ -dimensional cells in the momentum-space Grassmannian.

isolated points $C^* \in C \cap Z^\perp$, where Z is a *generic* configuration in $G(\mathbf{m}, n)$.

```
In[1]:= {intersectionNumber[{6,5,8,7,10,9,12,11}],
          intersectionNumber[{15,14,8,7,21,20,19,13,12,26,
                              25,24,18,17,31,30,29,23,22,36}]}

Out[1]:= {2,34}
```

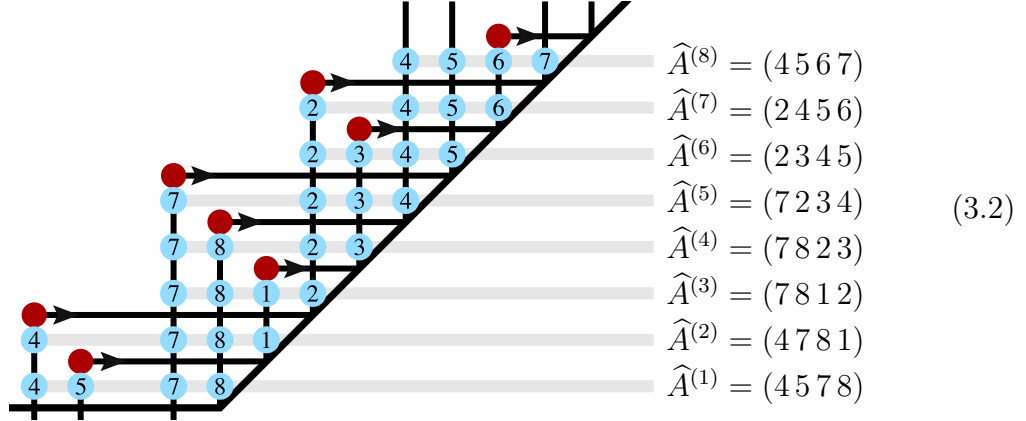
- `inverseBoundary[permutation_]`: returns a list of permutation labels for positroid cells in the *inverse*-boundary, ∂^{-1} , of the cell labeled by `permutation`.
- `legalPermQ[permutation_]`: tests whether the list `permutation` does in fact denote a (possibly decorated) permutation on n integers.
- `necklace[permutation_]`: if `permutation` labels a positroid configuration $C \in G(k, n)$, then `necklace` returns a list of n , k -tuples $A^{(a)} \equiv (A_1^{(a)}, \dots, A_k^{(a)})$ denoting the lexicographically-*minimal* non-vanishing minors starting from each column a . Recall from [1, 19] how the necklace encodes the list of all ranks of consecutive chains of columns of C ; for example, the necklace for the configuration in $G(4, 8)$ labeled by the permutation `{3, 7, 6, 10, 9, 8, 13, 12}` would be given by,

$$\begin{aligned}
A^{(8)} &= (8 \ 9 \ 10 \ 13) \\
A^{(7)} &= (7 \ 8 \ 9 \ 10) \\
A^{(6)} &= (6 \ 7 \ 9 \ 10) \\
A^{(5)} &= (5 \ 6 \ 7 \ 10) \\
A^{(4)} &= (4 \ 5 \ 6 \ 7) \\
A^{(3)} &= (3 \ 4 \ 5 \ 7) \\
A^{(2)} &= (2 \ 3 \ 4 \ 5) \\
A^{(1)} &= (1 \ 2 \ 4 \ 5)
\end{aligned} \tag{3.1}$$

This data is generated by the function `necklace` according to:

```
In[1]:= necklace[{3,7,6,10,9,8,13,12}]
Out[1]:= {{1,2,4,5},{2,3,4,5},{3,4,5,7},{4,5,6,7},
          {5,6,7,2},{6,7,1,2},{7,8,1,2},{8,1,2,5}}
```

- `necklaceR[permutation_]`: if `permutation` labels a positroid configuration $C \in G(k, n)$, then `necklace` returns a list of n , k -tuples $\hat{A}^{(a)} \equiv (\hat{A}_1^{(a)}, \dots, \hat{A}_k^{(a)})$ denoting the lexicographically-*maximal* non-vanishing minors starting from each column a . Like the more familiar ‘Grassmannian necklace’ generated by the function `necklace`, this data similarly encodes the ranks of all consecutive chains of columns of C ; to see this, consider the ‘reverse’ necklace for the configuration in $G(4, 8)$ labeled by the permutation $\{3, 7, 6, 10, 9, 8, 13, 12\}$,



This data is generated by the function `necklaceR` according to:

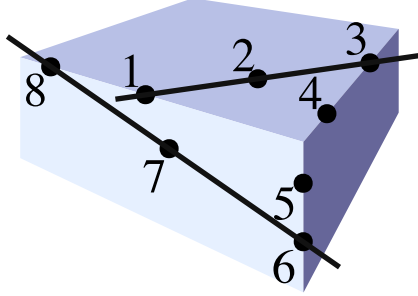
```
In[1]:= necklaceR[{3,7,6,10,9,8,13,12}]
Out[1]:= {{4,5,7,8},{1,4,7,8},{1,2,7,8},{2,3,7,8},
          {2,3,4,7},{2,3,4,5},{2,4,5,6},{4,5,6,7}}
```

- `nonSingularQ[permutation_,twistorDimension_:4]`: tests whether or not a configuration labeled by `permutation` has non-vanishing support for generic kinematical data—`nonSingularQ` returns `True` iff `intersectionNumber[permutation]>0`.
- `parityConjugate[permutation_]`: if `permutation` labels a configuration $C \in G(k, n)$, then `parityConjugate` returns the permutation labeling the geometrically dual configuration $C^\perp \in G(n-k, n)$. For example,

```
In[1]:= parityConjugate[{3,5,6,7,8,10}]
        parityConjugate@%
Out[1]:= {4,5,7,6,8,9}
        {3,5,6,7,8,10}
```


- `permToGeometry[permutation_,removeVanishingQ_>True]`: returns a (formatted) table of planes of various ranks spanned by consecutive chains of column-vectors of the configuration labeled by *permutation*; *distinguished* planes are highlighted in blue, and (for visual clarity) the option *removeVanishingQ* (`True` by default) causes any vanishing columns of the configuration to be suppressed.

For example, consider the positroid configuration labeled by the permutation $\{3, 7, 6, 10, 9, 8, 13, 12\}$,



consec. chains of columns	span	(3.3)
(1) (2) (3) (4) (5) (6) (7) (8)	\mathbb{P}^0	
(123) (34) (45) (56) (678) (81)	\mathbb{P}^1	
(56781) (81234) (3456)	\mathbb{P}^2	

For this, the function `permToGeometry` would produce:

```
In[1]:= permToGeometry[{3,7,6,10,9,8,13,12}]
Out[1]:= 
$$\begin{pmatrix} (1) (2) (3) (4) (5) (6) (7) (8) \\ (1\,2\,3) (3\,4) (4\,5) (5\,6) (6\,7\,8) (8\,1) \\ (3\,4\,5\,6) (5\,6\,7\,8\,1) (8\,1\,2\,3\,4) \end{pmatrix}$$

```

(Here, all the ‘distinguished’ maximal planes $(a+1, \dots, \sigma(a)-1)$ have been highlighted in blue—as the rest of the table follows from knowledge of these planes.)

- `permutationK[permutation_]`: returns the “ k ” associated with *permutation*. More explicitly, if $C \equiv (c_1, \dots, c_n)$ is the positroid configuration labeled by *permutation*, then `permutationK` returns $\text{rank}\{c_1, \dots, c_n\}$; as such, $C \in G(k, n)$. E.g.,

```
In[1]:= permutationK[{3,7,6,10,9,8,13,12}]
Out[1]:= 4
```

- `preferredGauge[permutation_]`: returns the lexicographically first non-vanishing minor of the configuration $C \in G(k, n)$; equivalently, if $C \equiv (c_1, \dots, c_n)$ is the configuration labeled by *permutation*, then `preferredGauge` returns the lexicographically-minimal set of column-labels (a_1, \dots, a_k) such that $\text{rank}\{c_{a_1}, \dots, c_{a_k}\} = k$.

- `randomCell[n_,k_,d_,exclusionsQ_:True]`: gives the permutation label of a randomly-generated, d -dimensional positroid cell $C \in G(k,n)$. If the optional argument `exclusionsQ` is `True` (its default value), then `randomCell` tries to find a configuration for which all columns are non-vanishing (when possible).
- `rotate[rotation_][permutation_]`: returns the permutation labeling a positroid cell whose columns are rotated (positively) relative to that of `permutation` by `rotation`; more specifically, given a positroid cell $C = (c_1, \dots, c_n) \in G(k, n)$ labeled by `permutation`, `rotate` returns the label of the configuration $C' = (c_{1'}, \dots, c_{n'})$ where $a' = a + \text{rotation}$.
- `storeBoundaries`: a global variable which by default is set to `False`; when `True`, boundary information is stored in memory so that boundary computations need not be repeated.

3.2 Positroid Coordinates and Matrix Representatives

- `bridgeToMinors[permutation_]`: using the BCFW-bridge construction of coordinates for the configuration C labeled by `permutation`, `bridgeToMinors` expresses each BCFW-bridge coordinate α_i directly in terms of the minors of C . For example, the (*canonical*) BCFW-bridge decomposition of the permutation `{3, 7, 6, 10, 9, 8, 13, 12}` generates the matrix representative:

$$\begin{array}{l} \text{In[1]:= permToMatrix}[\{3, 7, 6, 10, 9, 8, 13, 12\}] // \text{nice} \\ \text{Out[1]:= } \begin{pmatrix} 1 & \alpha_9 & 0 & -\alpha_5 & -\alpha_5 \alpha_6 & 0 & 0 & 0 \\ 0 & 1 & \alpha_8 & \alpha_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_3 + \alpha_6 & \alpha_3 \alpha_4 & 0 & -\alpha_1 \\ 0 & 0 & 0 & 0 & 1 & \alpha_4 & \alpha_2 & 0 \end{pmatrix} \end{array}$$

It is a highly non-trivial fact that such bridge coordinates α_i can be expressed as ratios of *monomials* involving the minors of C . The explicit correspondence is given by the function `bridgeToMinors`, as in the following example:

$$\begin{array}{l} \text{In[1]:= bridgeToMinors}[\{3, 7, 6, 10, 9, 8, 13, 12\}] // \text{nice} \\ \text{Out[1]:= } \begin{array}{lll} \alpha_1 \rightarrow \frac{(1258)}{(1245)} & \alpha_2 \rightarrow \frac{(1247)}{(1245)} & \alpha_3 \rightarrow \frac{(1245)(1278)(4567)}{(1246)(1247)(4578)} \\ \alpha_4 \rightarrow \frac{(1246)(1267)(4578)}{(1245)(1278)(4567)} & \alpha_5 \rightarrow \frac{(1246)(4578)}{(1278)(1456)} & \alpha_6 \rightarrow \frac{(1456)(2578)}{(1246)(4578)} \\ \alpha_7 \rightarrow \frac{(1478)}{(1278)} & \alpha_8 \rightarrow \frac{(1345)}{(1245)} & \alpha_9 \rightarrow \frac{(2456)}{(1456)} \end{array} \end{array}$$

- `matrixCharts[permutation_]`: gives a list of (distinct) matrix-representatives of the positroid configuration C labeled by `permutation`, using each of the (cyclically-distinct) columns as the ‘minimal’ column used in the construction of BCFW-bridge coordinates for C .
- `matrixToPerm[matrix_]`: returns the permutation which labels the positroid cell represented by `matrix`. If `matrix` is given in terms of unspecified variables, then `matrixToPerm` assumes that all such take on *generic* values.
- `permToMatrix[permutation_, transpositionScheme_ : 0]`: returns a matrix-representative of the positroid cell labeled by `permutation` given in terms of BCFW-bridge coordinates obtained using one of several possible bridge-decomposition schema—with the default scheme being “0”, that corresponding to the *canonical* or ‘lexicographic’ scheme described in reference [1]. The possible decomposition schema are those described for `transpositionChain`.

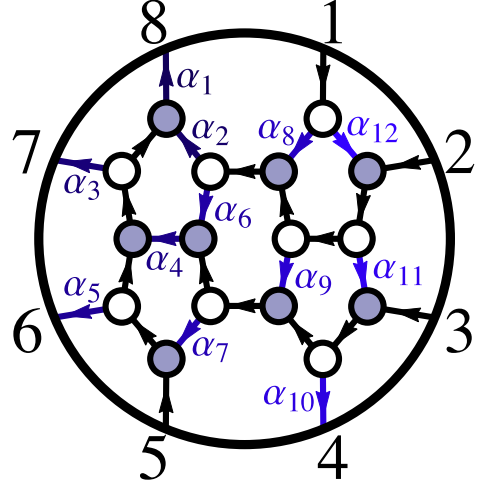
- `positiveQ[matrix_]`: returns `True` if the *all* ordered minors of the matrix are strictly non-negative; if *matrix* is parameterized by unspecified variables, then `positiveQ` returns `True` if *matrix* is positive (in the previous sense) when all its unfixed variables are given random, *positive* values.
- `transpositionChain[permutation_,scheme_:0]`: returns a list `{transpositionList, permutationList, seedGauge}` containing the complete BCFW-bridge decomposition of *permutation* into ‘adjacent’ transpositions according to the scheme denoted *scheme*. (This data is obviously redundant, but such redundancy proves somewhat useful to have at hand.) The possible schema include:

scheme	
0 (default)	the canonical or ‘lexicographic’ decomposition scheme
‘cyclic’	a scheme which attempts to decompose <i>permutation</i> into a sequence of bridges in a way which preserves cyclic symmetry (if any exists)
1	a scheme which <i>correctly orients</i> all $(2n-4)$ -dimensional cells in the momentum-space Grassmannian
-1	a scheme which <i>correctly orients</i> all $4k$ -dimensional cells in the momentum-twistor Grassmannian

To illustrate the different transposition schemes, the *lexicographic* scheme (“*scheme*=0”) would decompose the permutation $\{4, 7, 6, 9, 8, 11, 10, 13\}$,

“0” (Lexicographic) Bridge Decomposition Scheme

τ	1	2	3	4	5	6	7	8	BCFW shift
	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
(12)	4	7	6	9	8	11	10	13	$c_2 \mapsto c_2 + \alpha_{12} c_1$
(23)	7	4	6	9	8	11	10	13	$c_3 \mapsto c_3 + \alpha_{11} c_2$
(34)	7	6	4	9	8	11	10	13	$c_4 \mapsto c_4 + \alpha_{10} c_3$
(23)	7	6	9	4	8	11	10	13	$c_3 \mapsto c_3 + \alpha_9 c_2$
(12)	7	9	6	4	8	11	10	13	$c_2 \mapsto c_2 + \alpha_8 c_1$
(35)	9	7	6	4	8	11	10	13	$c_5 \mapsto c_5 + \alpha_7 c_3$
(23)	9	7	8	4	6	11	10	13	$c_3 \mapsto c_3 + \alpha_6 c_2$
(56)	9	8	7	4	6	11	10	13	$c_6 \mapsto c_6 + \alpha_5 c_5$
(35)	9	8	7	4	11	6	10	13	$c_5 \mapsto c_5 + \alpha_4 c_3$
(57)	9	8	11	4	7	6	10	13	$c_7 \mapsto c_7 + \alpha_3 c_5$
(25)	9	8	11	4	10	6	7	13	$c_5 \mapsto c_5 - \alpha_2 c_2$
(58)	9	10	11	4	8	6	7	13	$c_8 \mapsto c_8 + \alpha_1 c_5$
	9	10	11	4	13	6	7	8	

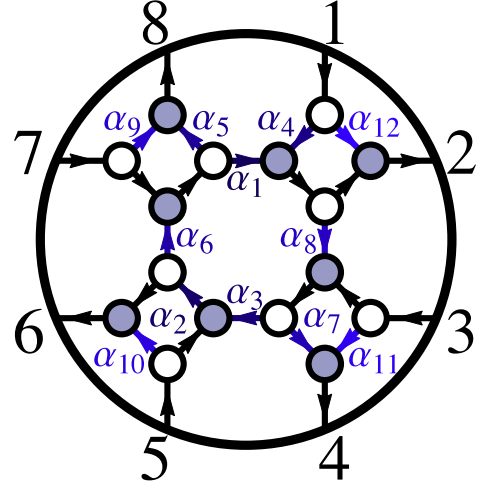


This decomposition would give rise to the following representative of the cell:

$$\begin{pmatrix} 1 & (\alpha_8 + \alpha_{12}) & (\alpha_9 + \alpha_{11})\alpha_8 & \alpha_8\alpha_9\alpha_{10} & 0 & 0 & 0 & 0 \\ 0 & 1 & (\alpha_6 + \alpha_9 + \alpha_{11}) & (\alpha_6 + \alpha_9)\alpha_{10} & (\alpha_6\alpha_7 - \alpha_2) & -\alpha_2\alpha_5 & -\alpha_2\alpha_3 & 0 \\ 0 & 0 & 1 & \alpha_{10} & (\alpha_4 + \alpha_7) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha_5 & \alpha_3 & -\alpha_1 \end{pmatrix} \quad (3.4)$$

However, the “cyclic” *scheme* would decompose the permutation according to:

“cyclic” Bridge Decomposition Scheme								
τ	1	2	3	4	5	6	7	8
	↓	↓	↓	↓	↓	↓	↓	↓
(12)	4	7	6	9	8	11	10	13
(34)	7	4	6	9	8	11	10	13
(56)	7	4	9	6	8	11	10	13
(78)	7	4	9	6	11	8	10	13
(23)	7	4	9	6	11	8	13	10
(34)	7	9	4	6	11	8	13	10
(67)	7	9	6	4	11	8	13	10
(78)	7	9	6	4	11	13	8	10
(12)	7	9	6	4	11	13	10	8
(35)	9	7	6	4	11	13	10	8
(56)	9	7	11	4	6	13	10	8
(72)	9	7	11	4	13	6	10	8
	9	2	11	4	13	6	15	8



which would result in the following matrix-representative:

$$\begin{pmatrix} 1 & (\alpha_4 + \alpha_{12}) & \alpha_4\alpha_8 & \alpha_4\alpha_8\alpha_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & (\alpha_7 + \alpha_{11}) & 0 & -\alpha_2 & -\alpha_2\alpha_6 - \alpha_2\alpha_6\alpha_9 & \\ 0 & 0 & 0 & 0 & 1 & (\alpha_3 + \alpha_{10}) & \alpha_3\alpha_6 & \alpha_3\alpha_6\alpha_9 \\ 0 & \alpha_1 & \alpha_1\alpha_8 & \alpha_1\alpha_8\alpha_{11} & 0 & 0 & 1 & (\alpha_5 + \alpha_9) \end{pmatrix} \quad (3.5)$$

3.3 Drawing On-Shell (Plabic) Graphs and Left-Right Paths

- `plabicGraph[permutation_, optionsList___:defaultOptions]`: draws a *reduced* plabic graph whose left-right path would be given by *permutation*. (Here, ‘plabic’ is used somewhat loosely, as the default behavior of `plabicGraph` is to draw graphs with monovalent and bivalent vertices when the inclusion of such is warranted.)

There are many possible options for `plabicGraph`; the principle of these are:

option		value	description
chainOption	★	0	uses the <i>lexicographic</i> bridge decomposition scheme
		“cyclic”	uses the bridge decomposition scheme “cyclic”
		± 1	uses the bridge decomposition scheme set by ± 1
rotation	★	0	constructs the graph using whatever bridge decomposition scheme is specified, but where particle ‘1’ is considered minimal
		a	same as above, but with cyclic ordering beginning with particle (1–a)
LRpaths	★	{}	does <i>not</i> show any left-right paths
		{a, ..., b}	draws the left-right paths which start at legs a, ..., b
		“All”	draws <i>all</i> left-right paths
directed	★	False	results in an <i>undirected</i> graph
		True	results in a <i>directed</i> graph whose <i>perfect orientation</i> follows from the BCFW bridge-decomposition scheme used
edgeLabels	★	False	does not label any edges of the graph
		True	labels the BCFW-bridge edges by the bridge coordinates associated with each
faceLabels	★	False	does not label any faces of the graph
		True	labels each face of the graph with a label “ f_i ”
		“A”	labels the faces of the graph with <i>A</i> -variable labels
		“X”	labels the faces of the graph with <i>X</i> -variable labels
showRemovable	★	True	shows monovalent vertices attached to legs which are self-identified under <i>permutation</i>
		False	removes boundary legs which are self-identified under <i>permutation</i>
orientation	★	1	draws the external legs with <i>clockwise</i> ordering
		-1	draws external legs with <i>counterclockwise</i> ordering
bipartiteQ	★	False	allows for trees of same-colored vertices
		True	collapses all same-colored trees giving a bipartite graph

Here, each **default** option has been marked with a “★”. In addition to these options, there are many which control stylistic and aesthetic details of the graphs generated by [plabicGraph](#); the principle among these include:

option		value	description
angle	★	0	draws the graph with particle 1 toward the middle-left
		θ	rotates the graph by θ radians in the <i>clockwise</i> direction
extLabels	★	“Auto”	labels the external legs $\{1, \dots, n\}$
		$\{a, \dots, b\}$	labels the external legs $\{a, \dots, b\}$
labelSpacing	★	0.65	places the external particle labels at a distance of 0.65 from the boundary
font	★	“Times”	uses the font “Times” for all labels
fontSize	★	36	sets the “FontSize” for external labels to be 36pt
imageSize	★	300	sets the “ImageSize” of the Graphics output to be 300
radius	★	4	sets the graph’s boundary as a circle with radius 4 units
vertexSize	★	0.325	draws all vertices with size of 0.325 units
lineThickness	★	3.5	draws edges with AbsoluteThickness [3.5]
LRpathThickness	★	4	draws left-right paths with AbsoluteThickness [4]
LRpathDistance	★	0.275	draws left-right paths with a distance of 0.275 units from the graph’s edges
LRarrowHeadSize	★	0.0655	sets the size of left-right path Arrowheads to be 0.0655
edgeArrowSize	★	0.05	sets the size of directed-edge Arrowheads to be 0.05
outerCircle	★	True	draws a disc at the boundary, enclosing the graph

Wherever sufficiently obvious, we have left as implicit the possible alternative settings allowed these options.

For a complete list of options for [plabicGraph](#)—together with the default value for each—consult the global variable “[defaultPlabicGraphOptions](#)”; users should find most detailed features of the output to be pliable through simple experimentation with the options.

- [plabicGraphData](#)[[permutation_](#),[transpositionScheme_](#):0]: given the [permutation](#) label for a positroid configuration a some [transpositionScheme](#), [plabicGraphData](#) returns the pair $\{\text{edgeList}, \text{faceList}\}$, where [edgeList](#) lists each (directed) edge $i \rightarrow j$ (carrying a weight w) as a triple $\{i, j, w\}$; and [faceList](#) lists the vertices along each face of the graph, listing them with clockwise ordering.
- [resetGraphDefaults](#): a function which automatically resets all the default graph options for the function [plabicGraph](#). (By default, [plabicGraph](#) remembers any options explicitly set until the length of permutations being drawn is changed.)

3.4 Physical Operations, Kinematics, and Scattering Amplitudes

- `bcfwPartitions[n_, k_]`: returns a list of pairs $\{(n_L, k_L), (n_R, k_R)\}$ which are bridged-together to compute the n -point $N^{(k-2)}$ MHV tree-amplitude $\mathcal{A}_n^{(k)}$ according to

$$\mathcal{A}_n^{(k)} = \sum_{\substack{(n_L, k_L) \\ (n_R, k_R)}} \mathcal{A}_{n_L}^{(k_L)} \bigotimes_{\text{BCFW}} \mathcal{A}_{n_R}^{(k_R)}. \quad (3.6)$$

- `bcfwTermNames[n_, k_]`: returns a *formatted* list of terms appearing in the BCFW tree-amplitude where MHV and $\overline{\text{MHV}}$ amplitudes have been collected together.

```
In[1]:= bcfwTermNames[6, 3]
bcfwTermNames[14, 7] [[9598]]
Out[1]:= {A5^(3) \otimes A3^(1), A4^(2) \otimes A4^(2), A3^(2) \otimes A5^(2)}
((A3^(2) \otimes (A4^(2) \otimes A4^(2))) \otimes A3^(1)) \otimes (A3^(2) \otimes ((A4^(2) \otimes A4^(2)) \otimes A3^(1)))
```

- `generalTreeContour[a_:=0, b_:=0, bridgeChoice_:=0][n_, k_]`: returns a list of permutation labels for the positroid cells which together give the n -particle $N^{(k-2)}$ MHV tree-amplitude obtained using the *white-to-black* BCFW-bridge attached to legs $(1\ n)$ for *bridgeChoice* ‘0’ (the default) or $(n\ 1)$ for *bridgeChoice* ‘1’, and for which the lower-point amplitudes appearing in the recursion have been recursed (using the same bridge-choice) attached to legs $(1+\mathbf{a}, n_L+\mathbf{a})$ and $(1+\mathbf{b}, n_R+\mathbf{b})$ of the left and right amplitudes, respectively.
- `identitySigns[permList_]`: returns a list of ± 1 of the relative signs needed for an identity involving the cells labeled by the permutations in *permList*. E.g.,

```
In[1]:= identitySigns[{{3, 2, 4, 5, 6, 7}, {2, 4, 3, 5, 6, 7}, {2, 3, 5, 4, 6, 7},
{2, 3, 4, 6, 5, 7}, {2, 3, 4, 5, 7, 6}, {1, 3, 4, 5, 6, 8}}]
Out[1]:= {1, -1, 1, -1, 1, -1}
```

- `nRatioContour[n_, k_]`: produces the same output as:

$$\text{permToResidue}/\text{@dualGrassmannian}/\text{@treeContour}[n, k].$$

- `nTreeContour[n_, k_]`: produces the same output as:

$$\text{permToResidue}/\text{@treeContour}[n, k].$$

- `permToResidue[permutation_]`: for a *permutation* labeling either a $(2n-4)$ -dimensional cell C in the momentum-space Grassmannian or a $4k$ -dimensional cell in the momentum-twistor Grassmannian, `permToResidue` uses the globally defined momentum twistors \mathbf{Zs} and the corresponding (or alternatively-defined) global spinor variables \mathbf{Ls} and \mathbf{Lbs} (λ and $\tilde{\lambda}$, respectively), to find the isolated point(s) $C^* \in C$ which solve the kinematical constraints and returns a pair (or list of pairs if more than one solution exists) $\{\mu^*, C^*\}$ where μ^* is the positroid measure evaluated at the point C^* which solves the kinematical constraints.

The on-shell function corresponding to a positroid cell labeled by σ , when evaluated at whatever kinematical data is given, would be given by,

$$f_\sigma = \mu^* \times \delta^{k \times 4}(C^* \cdot \tilde{\eta}), \quad \text{or} \quad f_\sigma = \mu^* \times \delta^{k \times 4}(C^* \cdot \eta), \quad (3.7)$$

if σ labels a cell in the momentum-space Grassmannian or the momentum-twistor Grassmannian, respectively.

- `positiveZs[nParticles_]`: it is sometimes convenient to evaluate analytic expressions involving spinor-helicity variables or momentum-twistors using explicit kinematical data; under such circumstances, there are some conveniences afforded by using “well-chosen” kinematical data.

Reasons for preferring one choice over another include: having all Lorentz invariants be integer-valued and relatively small; having all dual-conformal cross ratios *positive* (so as to avoid branch-ambiguities when evaluating the polylogarithms that arise in scattering amplitudes at loop-level); and possibly to have all Lorentz-invariants be distinct (either to help reconstruct an analytic expression or to avoid ‘accidental’ cancelations). Of these, the following momentum-twistors meet the first two desires spectacularly:

$$\mathbf{Zs} \equiv \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & \binom{n}{0} \\ 2 & 3 & 4 & 5 & \cdots & \binom{n+1}{1} \\ 3 & 6 & 10 & 15 & \cdots & \binom{n+2}{2} \\ 4 & 10 & 20 & 35 & \cdots & \binom{n+3}{3} \end{pmatrix}. \quad (3.8)$$

The function `positiveZs[16]` is evaluated when the `positroids` package is first loaded, allowing amplitudes involving as many as 16 particles to be evaluated without specific initialization.

- `randomTreeContour[n_, k_]`: returns the list of permutation labels for cells occurring in the BCFW tree-amplitude formula, where the lower-point amplitudes have been recursed using randomly-chosen legs.

- `setupUsingSpinors[lambdaList_,lambdaBarList_]`: sets up the global variables `Ls` and `Lbs` for λ and $\tilde{\lambda}$, respectively, and defines the global $(n \times 4)$ matrix `Zs` for momentum twistors for use in numerical evaluation.
- `setupUsingTwistors[twistorList_]`: sets up the global $(n \times 4)$ matrix `Zs` encoding the momentum-twistor kinematical data, and defines the auxiliary variables `Ls` and `Lbs` for λ and $\tilde{\lambda}$, respectively.
- `superComponent[component_][superFunction_]`: for the purposes of the `positroids` package, a *superFunction* must be given by a pair $\{f, C\}$ —an *ordinary* function $f(1, \dots, n)$ of the kinematical variables times a *fermionic* δ -function of the form,

$$\delta^{k \times 4}(C \cdot \tilde{\eta}) \equiv \prod_{I=1}^4 \left\{ \bigoplus_{a_1 < \dots < a_k} (a_1 \dots a_k) \tilde{\eta}_{a_1}^I \dots \tilde{\eta}_{a_k}^I \right\}, \quad (3.9)$$

where C is an $(n \times k)$ -matrix of ordinary functions, and for each $a = 1, \dots, n$, $\tilde{\eta}_a$ is a fermionic (anti-commuting) variable. To be clear, we consider each particle as a Grassmann coherent state of the form,

$$|a\rangle \equiv |a\rangle_{\{\}} + \tilde{\eta}_a^I |a\rangle_{\{I\}} + \frac{1}{2!} \tilde{\eta}_a^I \tilde{\eta}_a^J |a\rangle_{\{I,J\}} + \frac{1}{3!} \tilde{\eta}_a^I \tilde{\eta}_a^J \tilde{\eta}_a^K |a\rangle_{\{I,J,K\}} + \tilde{\eta}_a^1 \tilde{\eta}_a^2 \tilde{\eta}_a^3 \tilde{\eta}_a^4 |a\rangle_{\{1,2,3,4\}};$$

and if we use r_a to denote the R -charge of the a^{th} particle according to,

field	helicity	R -charge (r_a)	short-hand for r_a
$ a\rangle_{\{\}}$	+1	$\{\}$	p
$ a\rangle_{\{I\}}$	$+\frac{1}{2}$	$\{I\}$	p/2 ($\Leftrightarrow \{4\}$)
$ a\rangle_{\{I,J\}}$	0	$\{I, J\}$	—
$ a\rangle_{\{I,J,K\}}$	$-\frac{1}{2}$	$\{I, J, K\}$	m/2 ($\Leftrightarrow \{1, 2, 3\}$)
$ a\rangle_{\{1,2,3,4\}}$	-1	$\{1, 2, 3, 4\}$	m

then `superComponent[r_1, \dots, r_n][superFunction]` returns the *component* function of *superFunction* proportional to,

$$\prod_{a=1}^n \prod_{I \in r_a} \tilde{\eta}_a^I. \quad (3.10)$$

—that is, the component-function involving the states:

$$|1\rangle_{r_1} \dots |n\rangle_{r_n}. \quad (3.11)$$

For example, the “ $(-, +, -, +, -, +, -, +)$ ” component of the 8-particle N^2 MHV amplitude $\mathcal{A}_8^{(4)}$ proportional to,

$$(\tilde{\eta}_1^1 \tilde{\eta}_1^2 \tilde{\eta}_1^3 \tilde{\eta}_1^4)(\tilde{\eta}_3^1 \tilde{\eta}_3^2 \tilde{\eta}_3^3 \tilde{\eta}_3^4)(\tilde{\eta}_5^1 \tilde{\eta}_5^2 \tilde{\eta}_5^3 \tilde{\eta}_5^4)(\tilde{\eta}_7^1 \tilde{\eta}_7^2 \tilde{\eta}_7^3 \tilde{\eta}_7^4), \quad (3.12)$$

would be extracted by evaluating, (compare with e.g. [23, 24]):

<pre>In[1]:= Total[superComponent[m,p,m,p,m,p,m,p]/@ permToResidue/@treeContour[8,4]] Out[1]:= - 908 416 39 375</pre>
--

- `termsInBCFW[n_,k_,l_:0]`: gives the number of (non-vanishing) terms generated by the BCFW-recursion for the ℓ -loop, n -point $N^{(k-2)}$ MHV amplitude. (ℓ must be either 0 or 1 as the number of terms is scheme-dependent beyond 1-loop).

```
In[1]:= termsInBCFW[6,3]
        termsInBCFW[6,3,1]
Out[1]:= 3
        16
```

- `treeContour[n_,k_]`: returns the list of permutation labels for positroid cells which together give the n -particle $N^{(k-2)}$ MHV tree-amplitude, using the *default* recursion scheme; `treeContour[n,k]` is equivalent to `generalTreeContour[0,0,0][n,k]`.

```
In[1]:= treeContour[6,3]
Out[1]:= {{4,5,6,8, 7,9},{3,5,6,7,8,10},{4,6,5,7,8,9}}
```

3.5 Aesthetic and General Purpose Functions

- `explicitify[expression_, positiveQ_:=True]`: picks random (integers) for all variables occurring in an *expression*; if *positiveQ* is `True` (its default value), then the random assignments are taken to be positive. (If *expression* includes angle-brackets $\langle \dots \rangle$, `explicitify` will evaluate these assuming random kinematical data.)
- `exportToPDF[fileName_][figure_]`: saves a PDF version of *figure* to the file *fileName*⁴ using outlined fonts (and with other minor processing).
- `mod2[objectList_]`: returns the elements of *objectList* which occur an odd number of times in *objectList*. This can be useful, for example, if one wants to explicitly verify that $\partial^2 = 0 \pmod{2}$:

```
In[1]:= mod2[Join@@(boundary/@boundary[randomCell[8,4,12]])]
Out[1]:= {}
```

- `nice[expression_]`: formats *expression* to display ‘nicely’ by making replacements such as $\text{ab}[x \cdots y] \mapsto \langle x \cdots y \rangle$, $\alpha[1] \mapsto \alpha_1$, etc., and by writing any level-zero matrices in `MatrixForm`.
- `niceTime[timeInSeconds_]`: converts a time measured in seconds *timeInSeconds*, to human-readable form. For example,

```
In[1]:= niceTime[299 792 458]
         niceTime[3.1415926535]
Out[1]:= 9 years, 182 days
         3 seconds, 141 ms
```

- `random[objectList_]`: returns a random element from (the first level of) *objectList*.
- `randomSubset[subsetLength_][objectList_]`: returns a randomly-chosen subset of length *subsetLength* from among the list *objectList*.
- `timed[expression_]`: evaluates *expression* and prints a message regarding the time required for evaluation.

⁴The file is saved to the same directory as `Export[]`: either `NotebookDirectory[]` or `Directory[]` (if the former does not exist).

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